CLOCKS, ANGLES AND FUNCTIONS

Andy Kemp describes a week of 'timely' open-ended lessons with a high ability Y9 group of boys.[1]

This series of lessons arose from a chance encounter with a UKMT Intermediate Mathematical Challenge (IMC) paper and *GeoGebra*. During the previous week, the class had sat the IMC and in the following lesson I was going through some of the questions. One of the questions asked students to calculate the angle between the hands on a clock when the time was 08:06. The mathematics behind this question is not complicated but involves doing two calculations and then subtracting, which was difficult for me to do mentally while distracted by a barrage of other questions from the boys! As a result, when I went home that evening I sat down and spent a couple of minutes working out the answer on a scrap of paper and then recalled that I had recently been introduced to the fantastic open-source software *GeoGebra*, which is a dynamic geometry system with added algebraic elements. In an effort to understand the problem better and explain it to the boys (and as an excuse to experiment with using *GeoGebra*) I decided to try to build an animated clock.

I have described in the accompanying *webextra*[2] the process of building the clock, and the finished clock is also available.[3] Once the animated clock was finished I started to wonder what students could do with the idea of an analogue clock that might be mathematically interesting. One of the things I was particularly aiming for was a sense of purpose to the activity, something that made the exercise worthwhile.

The first lesson

I decided that a good starting point for the lessons would be to pick up on the ideas presented in the UKMT question but to ask the students to try to solve the more general problem. Their first task was to construct a function with two inputs (hours and minutes) which would output the angle between the hands. I felt this would give the students some sense of purpose, as they were trying to generate a mathematical entity to represent something they used regularly. I felt that this activity had the possibility of throwing up some of the students' misconceptions about angles (and functions) and would therefore hopefully give them an environment in which they could attempt to deal with these misconceptions.

I introduced the problem using my animated construction to remind them how the clock worked. I also provided an example for them to check their functions. After this brief introduction, the students worked in pairs for the remainder of the lesson (about 25 minutes).

Initially the students were unsure what it was that they were supposed to be doing and where to start. This, I felt, was probably a result of their inexperience tackling open-ended tasks (something I hope to rectify). After some gentle hints about considering the angles of each hand, some groups made progress whilst others took the less structured work as an excuse to try and get away with doing very little. This was a little disconcerting, as when asked to work on an exercise they are generally very focused and will remain on task for some time. I was concerned whether a more open-ended task was appropriate for these students and this is a point I will return to later.

As the students progressed, there was much discussion. Most groups spent quite a long time trying to work out how many degrees the minute and hour hands moved for each minute that passed – a task that I originally felt was quite simple. When the groups had been working for around 10 minutes I reminded everyone what the original task was. Some groups asked me to show them particular times on the clock so they could look at the angles involved. One group even got out a protractor and used it on the interactive whiteboard to find a value.

After groups had managed to calculate the movement per minute of each hand, they started thinking about the general situation. Interestingly, many of the groups tried to tackle both hands simultaneously, and as a result got confused about how to model the information. After some gentle hints about thinking of each hand separately, they started to generate expressions for each hand and looked at different ways of combining these expressions to find a function that generated the angle *between* the hands. I had an interesting discussion with one student about how we could make the function generate the *obtuse* angle instead of the *reflex* angle. He suggested a programming-like 'IF' statement which would check to see if the angle was over 180°. At this stage I would have liked to have been in a computer lab so that we could have explored his suggestion a little further.

I rounded off the lesson by drawing on the ideas from various groups and talking through the processes they had undertaken to reach their solutions.

I found this lesson to be interesting from an assessment point of view as it enabled me to see those who had a good understanding of angles and functions. However, for me the more interesting aspect was comparing those students who were able to tackle an open task with those who 'shut down'. I also noted which students quickly engaged with the task and which found it difficult to start.

The second lesson

The second lesson in the series took up a classic logic question related to clocks: 'How many times do the minute and hour hands on a clock cross in a 24 hour period?' Once again, I provided the students with little guidance beyond showing them the first couple of crossings using the animated *GeoGebra* clock.

Again, the approaches taken by the groups were diverse. Some groups used an analogue watch to explore the question and quickly reached an answer, although they were later confused as to how to justify their conclusion. Other groups made conjectures of various numbers and then explored whether or not they were true; eg, some groups started by thinking that the answer was one but quickly realised that this wasn't the case.

Some students looked at where the first crossing occurred to see if this might give some insight into the problem. I was interested to see how many of the groups seemed to be implementing strategies similar to those proposed by Polya (1945); eg, drawing diagrams, constructing equations and exploring sub-problems such as 'When do the hands first cross?' before approaching the more general problem. With hindsight, I wonder if suggesting those strategies in the first place might have helped those groups who seemed unable to begin. This is definitely an approach I will consider next time I set an open task.

As students started to reach sensible conclusions I asked them to consider two follow-on questions: 'What would change if there were 100 minutes/30 minutes/1 minute in an hour?' and 'What would happen if there were 10 hours/50 hours/1 hour in a day?' This was in an effort to help them explore what was always true for this question and what was reliant upon the particular time system involved. I found this to be a particularly useful section of the exercise. When summing up the activities of this lesson, this was one of the points I focused on. Again, with hindsight, I should have allowed more time for the plenary section to put together the various ideas that people had found. Looking back, I now realise that it would have been an interesting approach to try to use the function we had previously generated to solve this problem.

The third lesson

The third lesson in the series was a little bit more complicated. After thinking about how changes to the number of minutes in the hour might affect the previous question I decided to take the whole process one step further and convince the students that the European Union were intending to adopt 'Metric Time' from 1 January 2007. I mocked up a letter from the Headmaster (*figure 1*)[4] explaining the changes and stating that it would be the job of the mathematics department to explain how this new system of time would work.

After a little convincing, most of the students bought the idea and seemed to believe that it was really going to happen. I suggested that they might like to try to form some idea of how this new system would feel by working out what time certain things would happen under the new system; eg, getting up, school, lunch, dinner, bed time. It took the students a while to come to grips with the new system, with some confusion as to whether this meant the day was longer or whether the new second was the same length as the old second. Some groups spontaneously tried to work out the length of a metric second. As time progressed, I suggested they try to form a function that they could use to convert imperial time into metric time. At this point some students started to see the value of the metric time system and how much easier it was to work with the times. This was because they could form a single conversion that would give them the number of hours and the decimal places would give them minutes and seconds. There was discussion about what clocks would look like and some students set about designing a metric clock. One of the favourite points noted by the students was the fact that they would only be in school for less than 3 hours! At the end of the lesson, I had to let them know the truth that there were no immediate plans to implement 'metric time' but that it was a historical fact that it was tried briefly in France (1793-1795) and also goes by the name of 'French Revolutionary Time'.[5]

> Dear Student,
>
> **European Union to adopt Metric Time**
>
> As you may be aware, the European Union has decided, in an effort to aid scientists and computer programmers, to adopt a system of Metric Time (based upon the system previously know as French Revolutionary Time). This change is intended to begin on 1 January 2007.
>
> The important features of Metric Time are as follows:
> * 10 Metric Hours in a day
> * 100 Metric Minutes in a Metric Hour
> * 100 Metric Seconds in a Metric Minute
>
> Obviously the 'true' length of the day cannot change, so a metric second is not the same length as our normal second.
>
> I have asked the Mathematics department to distribute this information on my behalf, in the hope that they can better explain the implications and help you prepare effectively for the change next year.
>
> Yours sincerely,
>
> Headteacher

Overall I felt this lesson worked best out of the three. I put this down mostly to the students feeling there was a sense of purpose to it. Understanding metric time was something they really needed to be able to do and so this motivated them to explore time and the decimal system from a different perspective. Students seemed keen to explore avenues I had not suggested (eg, clock design) and were happy and engaged in the investigation for the whole duration of the lesson, apart from the occasional "You are kidding, aren't you sir?"

Conclusions

I was pleased with how some students engaged with an open-ended task but realise I need to develop further strategies for encouraging all students to engage – maybe this could involve giving them an outline for a particular investigation or perhaps more generic problem-solving advice in the form of some of Polya's strategies. I felt that the tasks didn't provide enough motivation for most of the students and I have decided to design a series of lessons in which the ultimate aim is for students to build a virtual clock in a dynamic geometry package. I feel this would require similar skills but might provide a more motivating environment.

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Notes

1 Set 2 of six in a selective independent school.

2 Go to www.atm.org.uk/mt/.

3 www.geogebra.at/en/wiki/index.php/English

4 Go to www.atm.org.uk/mt/ for a *Word* version of this letter, in case you want to adapt and use it with your students.

5 For another 'angle' on metric time, see *MT77*, 'Time goes metric', 15-16.

Reference

Polya, G. (1945) *How to Solve It*, 2nd edition, Princeton University Press

LETTER

Dear Editors,

Chris Brooksbank's letter (***MT197***, p45) raises a number of issues. The biggest one is 'Why do Geometry?', which deserves more than a one-line answer.

A lesser question was 'Why do glide-reflections?', a question which the articles by Wesslén and Fernandez (***MT191***, pp27-29) and Burke et al (***MT195***, pp12-14) addressed. Chris Brooksbank may have found their answers too abstract. The most familiar pattern which invites a description by glide-reflection is herring bone. Books on design are full of frieze patterns and borders. The chapter 'Repeating everything' in ***Symmetry a unifying concept*** by I. and M. Hargittai (Shelter Publications, 1994) has pictures enough to indicate a variety of practitioners with glide-reflections in their minds. A small collection of examples which evoke the glide-reflection description are in ***MT193*** in the bottom right-hand corner of page 33. Glide-reflections are for real!

Possibly Chris Brooksbank had not grasped the significance of the pentomino exercise in ***MT195*** on page 13. There is one and only one isometry taking Y → A, one taking Y → B, etc. The one taking Y → C is a glide-reflection. Much less obviously, the ones taking Y → H and Y → G are glide-reflections.

Bob Burn, Exeter